EULER-POINCARÉ EQUATIONS FOR NON-CONSERVATIVE ACTION PRINCIPLES

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Introduction

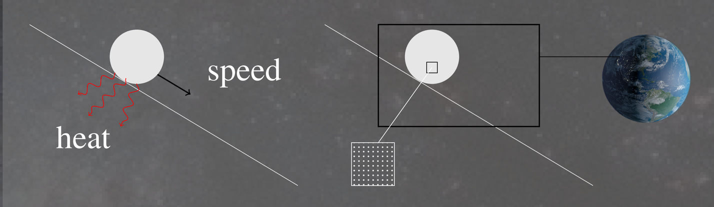
A fundamental principle in classical mechanics is Hamilton's principle, which states that the dynamics of a system are captured in a single action functional $\mathcal{S}$. However, for non-conservative physical laws time-symmetry is broken and Hamilton's principle is not valid.

Galley [1] developed a variational principle which allows to capture non-conservative interactions, providing equations of motion of the system in terms of Euler-Lagrange equations.
If a symmetry group is acting on the configuration space, the Euler-Lagrange equations can be reduced to Euler-Poincaré equations on its Lie algebra.

In order to formulate Galley's principle in coordinate free form, and to obtain deeper insight in the underlying geometry, we reformulate the principle in terms of deformations on a fluid manifold.

References:
[1] C. Galley (2013) *Classical Mechanics of Nonconservative Systems.*
[2] D. Holm, J. Marsden, and T. Ratiu (1999) *The Euler-Poincaré Equations in Geophysical Fluid Dynamics.*

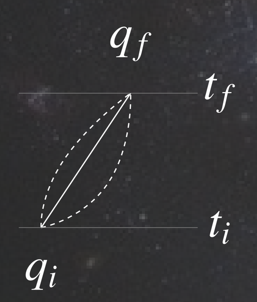
Nonconservative systems arise when only a subset of dynamical variables within a conservative system are considered.



Accessible and inaccessible degrees of freedom can be, for example, due to choice, obervational constraints, or separation of scales.

Hamilton's Principle

states that the dynamics of a physical system are determined by the variational problem: "Find a path from given intial value to a given final value, which makes the action $\mathcal{S}$ stationary."

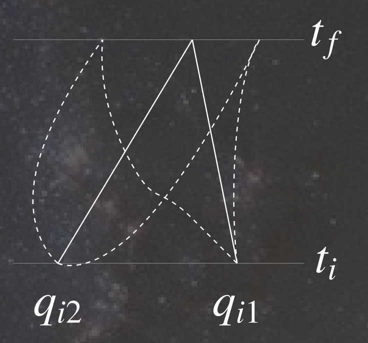


$$\delta\mathcal{S} = \delta\int_{t_i}^{t_f} L(q,\dot{q})\mathrm{d}t = 0$$

The Lagrangian L contains all physical interactions of the system. Typically, $L = T - V$, where T is the kinetic and V the potential energy.

Non-conservative Action Principle

It is well-known that Hamilton's principle is not applicable in non-conservative settings. By doubling the degrees of freedom,in [1] the system is expressed as a boundary value problem, leading to a similar action principle.



$$\mathcal{S} = \int_{t_i}^{t_f} \underbrace{L(q_1,\dot{q}_1)}_{\substack{\text{Lagrangian of } q_1\\ \text{integrated forward in time}}} - \underbrace{L(q_2,\dot{q}_2)}_{\substack{\text{Lagrangian of } q_2\\ \text{integrated backwards in time}}} + \underbrace{K(q_1,q_2,\dot{q}_1,\dot{q}_2)}_{\substack{\text{term coupling the variables } q_1, q_2\\ \text{'non-conservative potential'}}} \mathrm{d}t \tag{1}$$

The action is varied with the conditions that variations vanish at initial time and are equal (but not fixed) at final time. After the variation, setting $q_1 = q_2$ yields the real physical variable.

How does one choose the coupling term K?

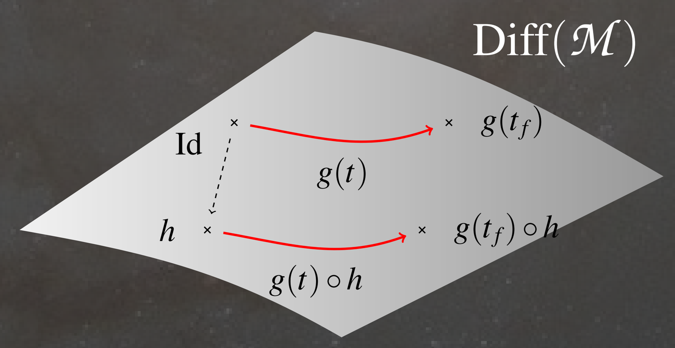
The 'non-conservative potential' K can for example be reconstructed from a known force on the system, or be obtained by integrating out inaccessible degrees of freedom.

Geometric Viewpoint

We consider fluid dynamics on a differentiable manifold $\mathcal{M}$, acted on by the group of diffeomorphisms $\mathrm{Diff}(\mathcal{M})$. The action $\mathcal{S}$ depends on paths $g \in \mathrm{Diff}(\mathcal{M}) \times [t_i, t_f]$ (which describes the evolution of fluid positions), $\dot{g}, \partial_x g$, and advected quantities $\phi_0 \in A$ (e.g., density, entropy), where A is a vector space.

Symmetry allows for a reduction of equations of motion. Let $\mathcal{S}$ be right-invariant under deformations $h \in \mathrm{Diff}(\mathcal{M})$, i.e. $\mathcal{S}(g) = \mathcal{S}(g \circ h) \forall h \in \mathrm{Diff}(\mathcal{M})$ for paths g. This is also referred to as relabeling symmetry.

Let $G := \mathrm{Diff}(\mathcal{M})$, and $\mathfrak{g}$ its Lie algebra.



Right-invariance of the Lagrangian $L\colon TG \times A \to \mathbb{R}$ allows the definition of a new Lagrangian $l\colon \mathfrak{g} \times A \to \mathbb{R}$ s.t.,

$$l(u,v,\phi) = L(g,\dot{g},\partial g,\phi_0) \tag{2}$$

with $u := \dot{g}g^{-1}, v := \partial_x g g^{-1}, \phi = \phi_0 g^{-1}$.

Euler-Poincaré equations resulting from varying $\int l(u,v,\phi)\mathrm{d}t$, are given in the Lagrangian (material) reference frame, in terms of the diffeomorphism and therefore coordinate-free. The Euler-Lagrange equations, resulting from varying (1), are given in Eulerian (spatial) coordinates.

Derivation of the Euler-Poincaré equations

We define l as in (2), and relate k to K analogously. For field theory we use the Lagrangian density ω, which satisfies $\int \omega \mathrm{d}x = w$ with

$$\begin{aligned} w(u_1,u_2,v_1,v_2,\phi_1,\phi_2) :=& l(u_1,v_1,\phi_1) - l(u_2,v_2,\phi_2) \\ &+ k(u_1,u_2,v_2,v_2,\phi_1,\phi_2). \end{aligned}$$

The goal is to compute variations $\delta\mathcal{S}$ for a variation of the deformation g.

Notation
$\mathcal{L}_u v$ Lie deriavtive of v wrt. u
$(\cdot,\cdot)$ dual pairing

Step 1: Compute variations δu, δv, $\delta\phi$, induced by a variaion δg. We express them depending on $\eta := \delta g g^{-1}$ and will obtain
$\delta u = \dot{\eta} + \mathcal{L}_u \eta \qquad \delta v = \dot{\eta} + \mathcal{L}_v \eta \qquad \delta\phi = -\mathcal{L}_\eta \phi$

Step 2: Insert this into the variation of the action

$$\delta\mathcal{S} = \int_{t_i}^{t_f} (\delta_u\omega|\delta u) + (\delta_v\omega|\delta v) \quad + (\delta_\phi\omega|\delta\phi)\mathrm{d}t,$$

Step 3: Formulate the above equation as a pairing with η. We use (i) Integration by parts on $\dot{\eta}$ and $\partial\eta$, (ii) $(\mu|\mathcal{L}_u\eta) = -(\mathcal{L}_u\mu|\eta)$, and (iii) $(\delta_\phi\omega \diamond \phi|\eta) := -(\delta_\phi\omega|\mathcal{L}_\eta\phi)$.

$$\begin{aligned} \delta\mathcal{S} =& -\int_{t_i}^{t_f} \Big(\underbrace{\frac{\mathrm{d}}{\mathrm{d}t}\delta_u\omega + \mathcal{L}_u\delta_u\omega + \partial\delta_v\omega + \mathcal{L}_v\delta_v\omega - \delta_\phi\omega \diamond \phi}_{=0} \Big| \eta \Big)\mathrm{d}t \\ & + \underbrace{\left[\int_{\mathcal{M}} \delta_u\omega\eta\mathrm{d}x\right]_{t_i}^{t_f} + \int_{t_i}^{t_f}\int_{\partial\mathcal{M}} \delta_v\omega\eta \cdot \mathrm{n}\mathrm{d}S(x)\mathrm{d}t}_{=0} \end{aligned}$$

Step 4: We choose boundary conditions such that the boundary term vanishes. Then by the fundamental lemma of variational calculus, we deduce the integrand is zero, i.e. we obtain the Euler-Poincaré equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\delta_u\omega + \mathcal{L}_u\delta_u\omega + \partial_x\delta_v\omega + \mathcal{L}_v(\delta_v\omega) = \delta_\Phi\omega \diamond \Phi$$

for the new antisymmetric Lagrangian ω.

Conclusion

We have

- formulated the action functional in terms of deformations of a fluid domain, to obtain a coordinate free-version
- used symmetry to define a Lagrangian on the Lie algebra
- found suitable boundary conditions for the doubled system
- obtained Euler-Poincaré equations by varying the new nonconservative action

Outlook Our goal is to obtain an expression for K by integrating out small scale fluctuations of the fluid flow. We want to use the Generalised Lagrangian Mean for a model of mean-fluctuation interactions which are compatible with the geometry and symmetry.